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SOME RESULTS ON THE NONSTATIONARY IDEAL II

BY

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ABSTRACT

This paper is a continuation of [Gi]. We show that the upper bound of [Gi] on the strength of NS_{μ^+} precipitous for a regular μ is exact. The upper bounds on the strength of NS_κ precipitous for inaccessible κ are reduced quite close to the lower bounds.

Introduction

The paper is a continuation of [Gi]. An understanding of [Gi] is required. However, there is one exception, Proposition 2.1. It does not require any previous knowledge and we think it is interesting on its own.

The paper is organized as follows: In Section 1 we examine the strength of NS_{μ^+} precipitous. The proof of the main theorem there is a continuation of the proof of 2.5.1 from [Gi]. Section 2 presents a proof of a "ZFC variant" of Lemma 2.18 of [Gi]. It was used in the previous version of this paper to deduce that saturatedness of $NS_\kappa^{\aleph_0}$ over an inaccessible κ implies an inner model with $\exists\alpha\ o(\alpha) = \alpha^{++}$. This was subsequently improved to inconsistency by S. Shelah and the author. In Section 3 a new forcing construction of NS_κ precipitous over inaccessible is sketched. It combines ideas from [Gi, Sec. 3] and [Gi1]. We assume familiarity with these papers.

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1. On the strength of precipitousness over a successor of regular

Our aim will be to improve the results of [Gi] on precipitousness of NS_{μ^+} for regular μ to the equiconsistency.* Throughout the paper $\mathcal{K}(\mathcal{F})$ is the Mitchell Core Model with the maximal sequence of measures $\mathcal{F}$, under the assumption $(\neg\exists\alpha\ o^{\mathcal{F}}(\alpha)=\alpha^{++})$. $o^{\mathcal{F}}(\kappa)$ denotes the Mitchell order of κ or, in other words, the length of the sequence $\mathcal{F}$ over κ. We refer to Mitchell [Mi1] for precise definitions.

In order to state the result let us recall a notion of (ω,δ)-repeat point introduced in [Gi].

Definition: Let α,δ be ordinals with $\delta < o^{\mathcal{F}}(\kappa)$. Then α is called a (ω,δ)-**repeat point** if (1) $\mathrm{cf}\,\alpha=\omega$, (2) for every $A\in\cap\{\mathcal{F}(\kappa,\alpha')|\alpha\le\alpha'<\alpha+\delta\}$ there are unboundedly many γ's in α such that $A\in\cap\{\mathcal{F}(\kappa,\gamma')|\gamma\le\gamma'<\gamma+\delta\}$.

We are going to prove the following:

THEOREM 1.1: *Suppose NS_{μ^+} is precipitous for a regular $\mu>\aleph_1$ and GCH. Then there exists an $(\omega,\mu+1)$-repeat point over μ^+ in $K(\mathcal{F})$.*

Remark: It is shown in [Gi] that starting with an $(\omega,\mu+1)$-repeat point it is possible to obtain a model of NS_{μ^+} precipitous. On the other hand, precipitousness of $NS^{\aleph_0}_{\mu^+}$ implies (ω,μ)-repeat point.

In what follows we will actually continue the proof of 2.5.1 of [Gi] and, assuming that the NS_{μ^+} is precipitous (or even only $NS^{\aleph_0}_{\mu^+}$ and $NS^{\mu}_{\mu^+}$), we will obtain $(\omega,\mu+1)$-repeat point.

Proof: Let $\kappa=\mu^+$. We consider the ordinal $\alpha^* < o^{\mathcal{F}}(\kappa)$ of the proof of 2.5.1 [Gi]. It was shown there to be a (ω,μ)-repeat point, under the assumption of nonexistence of up-repeat point and μ is not the successor of cardinal of cofinality ω. It was noted in [Gi] (the remark after Lemma 2.11) that if it is possible to remove the assumption of ω-closure of submodels in the Mitchell Covering Lemma, then the constructions of [Gi] apply also to μ which is the successor of a cardinal of cofinality ω. R.-D. Schindler claimed in [Sc] that the assumption on ω-closure can be removed. So, further, we do not separate the treatment of such μ's. Only submodels in this case, instead of being ω-closed, will be required to contain

* For a singular μ the situation is less clear but, recently, M. Magidor constructed a model with $NS_{\aleph_{\omega+1}}$ precipitous starting from a measurable Woodin cardinal. It appears close to equiconsistency by results of W. Mitchell, J. Steel and E. Schimmerling.

all implicitly mentioned ω-sequences. Intuitively, one can consider α^* as the least relevant ordinal. Basically, an ordinal α is called **relevant** if some condition in NS_κ forces that the measure $\mathcal{F}(\kappa,\alpha)$ is used first in the generic ultrapower to move κ and the cofinality of κ changes to ω. Using a nonexistence of up-repeat point, a set $A \in \mathcal{F}(\kappa,\alpha^*)$ such that $A \notin \mathcal{F}(\kappa,\beta)$ for β, $o^{\mathcal{F}}(\kappa) > \beta > \alpha^*$, was picked. This set A was used in [Gi] and will be used here to pin down α^*. Thus, for $\tau \le \kappa$, if there exists a largest $\tau_1 < o^{\mathcal{F}}(\beta)$ such that $A \cap \tau \in \mathcal{F}(\tau,\tau_1)$ then we denote it by τ^*. In this notation κ^* is just α^*. If $E = \{\tau < \kappa |$ there exists $\tau^*\}$ then $E \in \mathcal{F}(\kappa,\beta)$ for every β with $\alpha^* < \beta < o^{\mathcal{F}}(\kappa)$. Also, $A \cup E$ contains all points of cofinality ω of a club, since by the definition of α^*, $A \cup E \in \cap\{\mathcal{F}(\kappa,\alpha) | \alpha$ is a relevant ordinal$\}$.

CLAIM 1: The set of $\alpha < \kappa$ satisfying (a) and (b) below is stationary in κ.

(a) cf $\alpha = \mu$;

(b) for every $i < \mu$

$$\{\beta < \alpha \mid cf\ \beta = \aleph_0 \text{ and } o^{\mathcal{F}}(\beta) \ge \beta^* + i\}$$

is a stationary subset of α.

Proof: Otherwise, let C be a club avoiding all the α's which satisfy (a) and (b). Let N be a good model in the sense of 2.5.1 of [Gi], with $C \in N$. Consider $\langle \tau_n^N | n < \omega\rangle$, $\langle d_n^N | n < \omega\rangle$ and $\langle \beta_n^* | n < \omega\rangle$ of 2.5.1 [Gi]. Recall that $\langle \tau_n^N | n < \omega\rangle$ is a sequence of indiscernibles for N, each τ_n^N is a limit point of C, $d_n^N \subseteq C$ is an ω-club in $\bigcup(N \cap \tau_n)$ consisting of indiscernibles of cofinality ω in C, for $\nu \in d_n^N$ ν^* exists and β_n^* represents it over κ, i.e. $\nu^* = \mathbb{C}(\kappa, \beta_n^*, \beta(\nu))(\nu)$, where $\mathbb{C}$ is the coherence function (identically for every $\nu, \nu' \in d_n^N$). Also, for every $\tau < \tau'$ in d_n^N $\beta^N(\tau) < \beta^N(\tau')$, where $\beta^N(\tau)$ is the index of the measure on κ for which τ is an indiscernible.

Fix $n < \omega$. Then, $\tau_n \in C$. By 2.1 or 2.14 of [Gi] we can assume that cf $\tau_n = \mu$. Since (b) fails, there are $i_n < \mu$ and C_n a club of τ_n disjoint with

$$\{\nu < \tau_n | \operatorname{cf} \nu = \aleph_0 \text{ and } o^{\mathcal{F}}(\nu) \ge \nu^* + i_n\}.$$

Using elementarity of N, it is easy to find such C_n inside N. Let $\delta = \bigcup_{n<\omega} i_n$. Using 2.1.1 (or 2.15 for inaccessible μ) of [Gi] we will obtain $N^* \supseteq N$, which agrees (mod initial segment) with N about indiscernibles but has sets $d_n^{N^*}$ long enough to reach δ, i.e. there will be a final segment of τ's in $d_n^{N^*}$ with $\beta^{N^*}(\tau) > \beta_n^* + \delta$.

But then, for such τ, $o^{\mathcal{F}}(\tau) \geq \tau^* + \delta$. This is impossible, since C_n, $d_n^{N^*}$ are both clubs of τ_n in N^* with bounded intersection. Contradiction. ∎

Let S denote the set of α's satisfying the conditions (a) and (b) of Claim 1. Now form a generic ultrapower with S in the generic ultrafilter. Denote it by M and let $\mathcal{F}(\kappa,\xi)$ be the measure used to move κ. Then, in M $\mathrm{cf}\,\kappa = \mu$ and $S_i = \{\beta < \kappa \mid \mathrm{cf}\,\beta = \aleph_0 \text{ and } o^{\mathcal{F}}(\beta) > \beta^* + i\}$ is a stationary subset of κ for every $i < \mu$. Hence S_i is stationary also in V.

CLAIM 2: For every $i < \mu$ and $X \in \mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F})$, $X \in \mathcal{F}(\kappa, \alpha^* + i)$ iff $S_i \backslash \{\beta < \kappa | o^{\mathcal{F}}(\beta) > \beta^* + i \text{ and } X \cap \beta \in \mathcal{F}(\beta, \beta + i)\}$ is nonstationary.

Proof: Fix $i < \mu$. $\mathcal{F}(\kappa, \alpha^* + i)$ is an ultrafilter over $\mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F})$, so it is enough to show that for every $X \in \mathcal{F}(\kappa, \alpha^* + i)$ the set $S_i \backslash \{\beta < \kappa | o^{\mathcal{F}}(\beta) < \beta^* + i$ and $X \cap \beta \in \mathcal{F}(\beta, \beta + i)\}$ is nonstationary.

Suppose otherwise. Let $X \in \mathcal{F}(\kappa, \alpha^* + i)$ be so that

$$S' = S_i \backslash \{\beta < \kappa | o^{\mathcal{F}}(\beta) > \beta^* + i \text{ and } X \cap \beta \in \mathcal{F}(\beta, \beta + i)\}$$

is stationary.

Without loss of generality we may assume that S' already decides the relevant measure, i.e. for some $\gamma < o^{\mathcal{F}}(\kappa)$, S' forces the measure $\mathcal{F}(\kappa, \gamma)$ to be used first to move κ in the embedding into generic ultrapower restricted to $\mathcal{K}(\mathcal{F})$. Now, $S' \subseteq \{\beta < \kappa | o^{\mathcal{F}}(\beta) > \beta^* + i\}$. So, $\gamma > \gamma^* + i$, where γ^* is the largest ordinal γ^* below γ with $A \in \mathcal{F}(\kappa, \gamma^*)$. If $\gamma^* = \alpha^*$, then $\alpha^* + i < \gamma$ and hence $X^* = \{\beta < \kappa | o^{\mathcal{F}}(\beta) > \beta^* + i \text{ and } X \cap \beta \in \mathcal{F}(\beta, \beta + i)\} \in \mathcal{F}(\kappa, \gamma)$ since this is true in the ultrapower of $\mathcal{K}(\mathcal{F})$ by $\mathcal{F}(\kappa, \gamma)$. This leads to a contradiction, since, if $j : V \to M$ is a generic embedding forced by S', then $\kappa \in j(S')$ and $\kappa \in j(X^*)$, but $S' \cap X^* = \emptyset$. Contradiction.

If $\gamma^* < \alpha^*$, then also $\gamma < \alpha^*$ which is impossible, since there are no relevant ordinals below α^*. Also, γ^* cannot be above α^* since α^* is the last ordinal ξ with $A \in \mathcal{F}(\kappa, \xi)$. ∎

For $i < \mu$ and a set $X \subseteq \kappa$ let us denote by X_i^* the set

$$\{\beta < \kappa | o^{\mathcal{F}}(\beta) > \beta^* + i \text{ and } X \cap \beta \in \mathcal{F}(\beta, \beta + i)\}.$$

By Cub_κ we denote the closed unbounded filter over κ and let $\mathrm{Cub}_\kappa \restriction S_i$ be its restriction to S_i, i.e. $\{E \subseteq \kappa | E \supseteq C \cap S_i \text{ for some } C \in \mathrm{Cub}_\kappa\}$.

CLAIM 3: For every $i < \mu$,

$$\mathcal{F}(\kappa, \alpha^* + i) = \{X \in (\mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F}))^M | X_i^* \in (\mathrm{Cub}_\kappa \restriction S_i)^M\}.$$

Proof: Let $X \in \mathcal{F}(\kappa, \alpha^* + i)$; then, by Claim 2, $X_i^* \in \mathrm{Cub}_\kappa \restriction S_i$ in V. But then, also in M, $X_i^* \in (\mathrm{Cub}_\kappa \restriction S_i)^M$, since $(\mathrm{Cub}_\kappa)^M \supseteq (\mathrm{Cub}_\kappa)^V$. Now, if $X \notin \mathcal{F}(\kappa, \alpha^* + i)$, then $Y = \kappa \backslash X \in \mathcal{F}(\kappa, \alpha^* + i)$, assuming $X \in \mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F})$. By the above, $Y_i^* \in (\mathrm{Cub}_\kappa \restriction S_i)^M$. But $X \cap Y = \emptyset$ implies $X_i^* \cap Y_i^* = \emptyset$. So $X_i^* \notin (\mathrm{Cub}_\kappa \restriction S_i)^M$. ∎

CLAIM 4: $o^{\mathcal{F}}(\kappa) > \alpha^* + \mu$.

Proof: By Claim 3, $\mathcal{F}(\kappa, \alpha^* + i) \in M$ for every $i < \mu$. Hence $(o(\kappa))^M \geq \alpha^* + \mu$. But now, in V, $o^{\mathcal{F}}(\kappa) \geq \alpha^* + \mu + 1$. ∎

We actually showed more:

CLAIM 5: $S \Vdash$ " $\underset{\sim}{\xi} \geq \alpha^* + \mu$ and for every $i < \mu$

$$\mathcal{F}(\kappa, \alpha^* + i) = \{X \in (\mathcal{P}(\kappa) \cap \mathcal{K}(\mathcal{F}))^{\underset{\sim}{M}} | X_i^* \in (\mathrm{Cub}_\kappa \restriction S_i)^{\underset{\sim}{M}}\}",$$

where $\underset{\sim}{\xi}$ is a name of the index of the first measure $\mathcal{F}(\kappa, \xi)$ used to move κ and $\underset{\sim}{M}$ is a generic ultrapower.

In order to complete the proof, we need to show that every $Y \in \mathcal{F}(\kappa, \alpha^* + \mu)$ belongs to $\mathcal{F}(\kappa, \gamma)$ for unboundedly many γ's below α^*. The conclusion of the theorem will then follow by [Gi, Sec. 1]. So let $Y \in \mathcal{F}(\kappa, \alpha^* + \mu)$. Consider the set $Y^* = \{\beta < \kappa \mid \beta^* \text{ exists}, o^{\mathcal{F}}(\beta) > \beta^* + \mu \text{ and } Y \cap \beta \in \mathcal{F}(\kappa, \beta^* + \mu)\} \cup Y$. Then $Y^* \in \cap\{\mathcal{F}(\kappa, \alpha) | \alpha^* + \mu \leq \alpha < o^{\mathcal{F}}(\kappa)\}$. It is enough to show that Y^* belongs to $\mathcal{F}(\kappa, \gamma)$ for unboundedly many γ's below α.

CLAIM 6: *$S \backslash Y^*$ is nonstationary.*

Proof: Suppose otherwise. Let $S' \subseteq S \backslash Y^*$ be a stationary set forcing $\mathcal{F}(\kappa, \xi)$ to be the first measure used to move κ in the ultrapower, where $\xi < o^{\mathcal{F}}(\kappa)$. Then, by Claim 5, $\xi \geq \alpha^* + \mu$. Hence, $Y^* \in \mathcal{F}(\kappa, \xi)$, which is impossible, since $Y^* \cap S' = \emptyset$. Contradiction. ∎

CLAIM 7: *α^* is a $\mu + 1$-repeat point.*

Proof: Let Y^* be as above. It is enough to find $\gamma < \alpha^*$ such that $Y^* \in \mathcal{F}(\kappa, \gamma)$. Let $C \subseteq \kappa$ be a club avoiding $S \backslash Y^*$. Let N, $\{\tau_n | n < \omega\}$ be as in the proof of

Claim 1 (i.e. as in the proof of 2.5.1 [Gi]) only with the club of Claim 1 replaced by C and with $Y^* \in N$. Then τ_n's are in $S \cap C$, and hence in Y^*, which means that for all but finitely many n's, $Y^* \in \mathcal{F}(\kappa, \beta^N(\tau_n))$, by [Mi1 ,Mi2], since τ_n's are indiscernibles for $\beta^N(\tau_n)$'s. ∎

The claim does not rule out the possibility that some Y^* reflects only boundedly many times below α^*. Thus, there is possibly some $\eta < \alpha^*$ such that the $\beta^N(\tau_n)$'s of Claim 7 are always below η. This would mean that $\beta_n^* > \beta^N(\tau_n)$, where β_n^* is the stabilized value of $(\beta(\nu))^*$ for $\nu \in d_n^N$. We will use Claim 5 in order to show that this is impossible. Namely, the following holds:

Claim 8: In the notation of Claim 7, for all but finitely many n's, $\left(\beta^N(\tau_n)\right)^* = \beta_n^*$.

Proof: By Claim 5, for all but nonstationary many ν's in S the following property $(*)$ holds: $o^{\mathcal{F}}(\nu) \geq \nu^* + \mu$ and, for every $i < \mu$, $\mathcal{F}(\nu, \nu^* + i) = \left\{X \in \mathcal{P}(\nu) \cap \mathcal{K}(\mathcal{F}) | X_i^* \in \mathrm{Cub}_\nu \restriction \{\rho < \nu | cf \rho = \aleph_0 \text{ and } o^{\mathcal{F}}(\rho) > \rho^* + i\}\right\}$.

Without loss of generality let us assume that $(*)$ holds for every element of S, otherwise just remove the nonstationary many points. Then, preserving notations of Claim 7, τ_n's satisfy $(*)$. We now show that ultrafilters $\mathcal{F}(\tau_n, \tau_n^* + i)$ correspond to $\mathcal{F}(\kappa, \beta_n^* + i)$ (i.e. $\tau_n^* + i = \mathbb{C}(\kappa, \beta_n^* + i, \beta(\tau_n))(\tau_n)$) for all but finitely many $n < \omega$ and all $i < \mu$.

Let $\overline{\beta}_n^*$ denote $\left(\beta^N(\tau_n)\right)^*$ and we will drop the upper index N further. Then $\tau_n^* + i = \mathbb{C}(\kappa, \overline{\beta}_n^* + i, \beta(\tau_n))(\tau_n)$ for every $n < \omega$, where $\mathbb{C}$ is the coherence function (see [Mi1] or [Gi]). Suppose that $\beta_n^* \neq \overline{\beta}_n^*$ for infinitely many n's. For simplicity let us assume that this holds for every $n < \omega$. In the general case only the notation is more complicated. There will be $X_n \in \left(\mathcal{F}(\kappa, \overline{\beta}_n^*) \backslash \mathcal{F}(\kappa, \beta_n^*)\right) \cap N$ for every $n < \omega$, since N is an elementary submodel. Let $n < \omega$ be fixed. Pick $\mathcal{K}(\mathcal{F})$-least $X_n \in \mathcal{F}(\kappa, \overline{\beta}_n^*) \backslash \mathcal{F}(\kappa, \beta_n^*)$. Still it is in N by elementarity. Also its support (in the sense of [Mi1, Mi2]) will be below τ_n, i.e. $X_n = h^N(\delta)$, for $\delta < \tau_n$, where h^N is the Skolem function of $N \cap \mathcal{K}(\mathcal{F})$. The reason for this is that X_n appears once both $\overline{\beta}_n^*$ and β_n^* appear. But $\overline{\beta}_n^*$ appear below τ_n since the support of τ_n is below τ_n and β_n^* appear before τ_n since, for $\nu \in d_n \subseteq \tau_n$, $\left(\beta^N(\nu)\right)^* = \beta_n^*$. Hence $X_n \cap \tau_n \in \mathcal{F}(\tau_n, \tau_n^*)$. Then by $(*)$,

$$(X_n)_0^* \in \mathrm{Cub}_{\tau_n} \restriction \{\rho < \tau_n | \operatorname{cf} \rho = \aleph_0 \text{ and } o^{\mathcal{F}}(\rho) > \rho^*\}.$$

This is clearly true also in N. But then $(X_n)_0^* \cap \cup(N \cap \tau_n)$ contains an ω-club intersected with the set $\{\rho < \tau_n | \operatorname{cf} \rho = \aleph_0 \text{ and } o^{\mathcal{F}}(\rho) > \rho^*\}$. Hence $(X_n)_0^* \cap d_n$

is unbounded in $\bigcup(N \cap \tau_n)$. Then $(X_n)_0^* \in \mathcal{F}(\kappa, \beta_n^* + i)$ for some i, $0 < i < \mu$, which implies that $X_n \in \mathcal{F}(\kappa, \beta_n^*)$. Contradiction. ∎

Combining Claims 7 and 8 we obtain that $Y^* \in \mathcal{F}(\kappa, \beta_n^* + \chi)$ for some $\chi \geq \mu$, for all but finitely many n's. Now, β_n^*'s are unbounded in α^* by [Gi] and hence we have an unbounded reflection of Y below α^*. ∎

2. On a fast sequence of ordinals or "ZFC variant" of a lemma of [Gi]

In this section were present a "ZFC variant" of Lemma 2.18 of [Gi]. It was used here originally to answer a question of [Gi] showing that the saturatedness of $NS_\kappa^{\aleph_0}$ over inaccessible κ implies an inner model with $\exists\alpha\ o(\alpha) = \alpha^{++}$. But since then it was shown by S. Shelah and the author that $NS_\kappa^{\aleph_0}$ cannot be saturated over an inaccessible κ. We think that this "ZFC variant" is still interesting. Moreover a variation of it turned out to be crucial in the proof of the inconsistency. The argument here will be somewhat simpler and for a reader familiar with generic ultrapowers it will be easy to relate it to saturated ideals.

PROPOSITION 2.1: *Let $V_1 \subseteq V_2$ be two models of ZFC. Let κ be a regular cardinal of V_1 which changes its cofinality to Θ in V_2. Suppose that in V_1 there is an almost decreasing (mod nonstationary or equivalently mod bounded) sequence of clubs of κ of length $(\kappa^+)^{V_1}$ so that every club of κ of V_1 almost contains one of the clubs of the sequence. Assume that V_2 satisfies the following:*

(1) $\mathrm{cf}(\kappa^+)^{V_1} \geq (2^\Theta)^+$ *or* $\mathrm{cf}(\kappa^+)^{V_1} = \Theta$;

(2) $\kappa > \Theta^+$.

Then in V_2 there exists a cofinal in κ sequence $\langle \tau_i \mid i < \Theta\rangle$ consisting of ordinals of cofinality $\geq \Theta^+$ so that every club of κ of V_1 contains a final segment of $\langle \tau_i \mid i < \Theta\rangle$.

Remark: (1) If in V_1, $2^\kappa = \kappa^+$, then clearly there exists an almost decreasing sequence of clubs of κ of length κ^+ so that every club of κ of V_1 almost contains one of the clubs of the sequence.

(2) M. Dzamonja and S. Shelah [D-Sh] using club guessing techniques were able to replace the condition (1) by weaker conditions.

Proof: If $\mathrm{cf}(\kappa^+)^{V_1} = \Theta$ then we can simply diagonalize over all the clubs. So let us concentrate on the case $\mathrm{cf}(\kappa^+)^{V_1} \geq (2^\Theta)^+$. Suppose otherwise. Assume for simplicity that $\Theta = \aleph_0$. Let C be a club in κ in V_1. Define in V_2 a wellfounded tree

$\langle T(C), \leq_C\rangle$. Fix a well ordering $\prec$ of a larger enough portion of V_2. Let the first level of $T(C)$ consist of the $\prec$-least cofinal in κ sequence of order type ω. Suppose that $T(C) \restriction n+1$ is defined. We define $\mathrm{Lev}_{n+1}(T(C))$. Let $\eta \in \mathrm{Lev}_n(T(C))$. Let η^* be the largest ordinal in $T(C) \restriction n+1$ below η. We assume by induction that it exists. If $\mathrm{cf}\,\eta = \aleph_0$, then pick $\langle \eta_n \mid n < \omega\rangle$ the least cofinal sequence in η of order type ω. Let the set of immediate successors of η, $\mathrm{Suc}_{T(C)}(\eta)$, be $\{\eta_n \mid n < \omega,\ \eta_n > \eta^*\}$.

If $\mathrm{cf}\,\eta \geq \aleph_1$, then consider $\eta' = \bigcup(C \cap \eta)$. If $\eta' = \eta$, then let $\mathrm{Suc}_{T(C)}(\eta) = \emptyset$. If $\eta^* < \eta' < \eta$, then let $\mathrm{Suc}_{T(C)}(\eta) = \{\eta'\}$. Finally, if $\eta' \leq \eta^*$ then let $\mathrm{Suc}_{T(C)}(\eta) = \emptyset$. This completes the inductive definition of $\langle T(C), \leq_C\rangle$. Obviously, it is wellfounded and countable. Let $T^*(C)$ denote the set of all endpoints of $T(C)$ which are in C. Notice that by the construction any such point is of uncountable cofinality. Also, $T^*(C)$ is unbounded in κ, since $\mathrm{otp}(C) = \kappa$ and $\kappa > \aleph_1$.

There must be a club $C_1 \subseteq C$ in V_1 avoiding unboundedly many points of $T^*(C)$, since otherwise the sequence $\langle \tau_i | i < \aleph_0\rangle$ required by the proposition could be taken from $T^*(C)$. This means, in particular, that for every $\alpha < \kappa$ there will be

$$\bar{\nu} = \langle \nu_1, \ldots, \nu_n\rangle \in T(C) \cap T(C_1)$$

so that

(a) $\mathrm{cf}\,\nu_n > \aleph_0$;

(b) $\mathrm{Suc}_{T(C)}(\nu_n) = \{\nu_{n+1}\}$ for some $\nu_{n+1} \in C \backslash \alpha$;

(c) either

(c1) $\mathrm{Suc}_{T(C_1)}(\nu_n) = \emptyset$

or

(c2) for some $\rho \in (C_1 \cap \nu_{n+1}) \backslash \alpha$ $\mathrm{Suc}_{T(C_1)}(\nu_n) = \{\rho\}$.

Now define a sequence $\langle C_\alpha \mid \alpha < (2^{\aleph_0})^+\rangle$ of clubs so that

(1) C_α is a club in κ in V_1;

(2) if $\beta < \alpha$ then $C_\alpha \backslash C_\beta$ is bounded in κ;

(3) $C_{\alpha+1}$ avoids unboundedly many points of $T^*(C_\alpha)$.

Since $\mathrm{cf}(\kappa^+)^{V_1} \geq (2^{\aleph_0})^+$ and in V_1 there is an almost decreasing (mod bounded) sequence of κ^+-clubs generating the club filter, there is no problem in carrying out the construction of $\langle C_\alpha \mid \alpha < (2^{\aleph_0})^+\rangle$ satisfying (1)–(3). The construction of $C_{\alpha+1}$ over C_α will be like those above for C_1 and C. Also the conditions (a), (b), (c) above will be satisfied by C_α, $C_{\alpha+1}$ replacing C, C_1.

Shrinking the set of α's if necessary we can assume that for every $\alpha, \beta < (2^{\aleph_0})^+$ $\langle T(C_\alpha), \leq_{C_2}, \leq\rangle$ and $\langle T(C_\beta), \leq_{C_\beta}, \leq\rangle$ are isomorphic as trees with ordered levels.

Let $\langle \kappa_m \mid m < \omega\rangle$ be the $\prec$-least cofinal in κ sequence.

Let $\alpha < \beta < (2^{\aleph_0})^+$. Since C_β is almost contained in $C_{\alpha+1}$, it avoids unboundedly many points in $T^*(C_\alpha)$. So for every $m < \omega$ there is $\bar{\nu} = \langle \nu_1, \ldots, \nu_n\rangle \in T(C_\alpha) \cap T(C_\beta)$ so that

(a) $\mathrm{cf}\, \nu_n > \aleph_0$;

(b) $\mathrm{Suc}_{T(C_\alpha)}(\nu_n) = \{\nu^\alpha_{n+1}\}$ for some $\nu^\alpha_{n+1} \in C_\alpha \backslash \kappa_m$;

(c) for some $\nu^\beta_{n+1} \in (C_\beta \cap \nu^\alpha_{n+1})\backslash \kappa_m$, $\mathrm{Suc}_{T(C_\beta)}(\nu_n) = \{\nu^\beta_{n+1}\}$.

Thus, pick $\ell > m$ so that $C_\beta \backslash \kappa_{\ell-1} \subseteq C_\alpha$. We consider subtrees

$$T(C_\gamma)_\ell = \{\bar{\eta} \in T(C_\gamma) | \exists k \geq \ell\ \bar{\eta}\ \geq_{C_\gamma} \langle \kappa_k\rangle\}$$

where $\gamma = \alpha, \beta$.

Let π be an isomorphism between $T(C_\alpha)$ and $T(C_\beta)$ respecting the order of the levels. Notice that the first level in both trees is the same $\{\kappa_i | i < \omega\}$. Hence, π will move $T(C_\alpha)_\ell$ onto $T(C_\beta)_\ell$.

Pick the maximal $n < \omega$ such that π is an identity on $\big(T(C_\alpha)_\ell\big) \restriction n+1$. It exists since $T^*(C_\alpha)\backslash C_\beta$ is unbounded in κ. Now let ν be the least ordinal in $\mathrm{Lev}_{n+1}\big(T(C_\alpha)_\ell\big)$ such that $\pi(\langle \nu_1, \ldots, \nu_n, \nu\rangle) \neq \langle \nu_1, \ldots, \nu_n, \nu\rangle$, where $\langle \nu_1, \ldots, \nu_n\rangle$ is the branch of $T(C_\alpha)_\ell$ leading to ν.

Consider ν_n. If $\mathrm{cf}\, \nu_n = \aleph_0$, then we are supposed to pick the $\prec$-least cofinal in ν_n sequence $\langle \nu_{ni} | i < \omega\rangle$ and the maximal element ν^*_n of the tree $T(C_\alpha)$ below ν_n. $\mathrm{Suc}_{T(C_\alpha)}(\nu_n)$ will be $\{\nu_{ni} | i < \omega \text{ and } \nu_{ni} > \nu^*_n\}$. Notice that $\nu^*_n \geq \kappa_{n-1}$ by the definition of the tree $T(C_\alpha)$. Hence, either $\nu^*_n = \kappa_{n-1}$ or $\nu^*_n \in T(C_\alpha)_\ell \restriction n+1$ since elements of $T(C_\alpha)$ which are above κ_{n-1} in the tree order are below it as ordinals. But since $T(C_\alpha)_\ell \restriction n+1 = T(C_\beta)_\ell \restriction n+1$ and $\kappa_{\ell-1} \in T(C_\beta)$, the same is true about $\mathrm{Suc}_{T(C_\beta)}(\nu_n)$, i.e. it is $\{\nu_{ni} | i < \omega \text{ and } \nu_{ni} > \nu^*_n\}$. Then π will be an identity on $\mathrm{Suc}_{T(C_\alpha)}(\nu_n)$ and, in particular, will not move ν. Contradiction.

So $\mathrm{cf}\, \nu_n$ should be above $\aleph_0$. Once again the maximal elements of $T(C_\alpha) \restriction n+1$ and $T(C_\beta) \restriction n+1$ below ν_n are the same. Let ν^*_n denote this element. Now, $\nu \in \mathrm{Suc}_{T(C_\alpha)}(\nu_n)$, hence $\nu = \bigcup(C_\alpha \cap \nu_n)$ $\nu^*_n < \nu < \nu_n$ and $\mathrm{Suc}_{T(C_\alpha)}(\nu_n) = \{\nu\}$ by the definition of the tree $T(C_\alpha)$. π is an isomorphism, so $\mathrm{Suc}_{T(C_\beta)}(\nu_n) \neq \emptyset$. By the definition of the tree $T(C_\beta)$, $\nu^* < \nu' < \nu_n$ and $\mathrm{Suc}_{T(C_\beta)}(\nu_n) = \{\nu'\}$ where

$\nu' = \bigcup(C_\beta \cap \nu_n)$. By the choice of ν, $\nu \neq \nu'$. But $\nu, \nu' > \kappa_{\ell-1}$ and $C_\beta \backslash \kappa_{\ell-1} \subseteq C_\alpha$, so $\nu' \in C_\alpha$. Hence $\nu' < \nu$ and the sequence $\langle \nu_1, \ldots, \nu_n \rangle$ is as desired.

Let $\langle T, \leq_T, \leq \rangle$ be a countable tree consisting of countable ordinals with the usual order $\leq$ between them isomorphic to $\langle T(C_\alpha), \leq_{C_\alpha}, \leq \rangle$ $\left(\alpha < (2^{\aleph_0})^+\right)$. Define a function $h : [(2^{\aleph_0})^+]^2 \to \omega$ as follows: $f(\alpha, \beta)$ = the minimal element of T corresponding to some

$$\overline{\nu} = \langle \nu_1, \ldots, \nu_n \rangle \in T(C_\alpha) \cap T(C_\beta)$$

satisfying the conditions (a), (b) and (c).

By Erdös–Rado there exists a homogeneous infinite set $A \subseteq (2^{\aleph_0})^+$. Let $\langle \alpha_n \mid n < \omega \rangle$ be an increasing sequence from A. Then there is $\overline{\nu} = \langle \nu_1, \ldots, \nu_n \rangle \in \bigcap_{m<\omega} T(C_{\alpha_m})$ witnessing (a), (b), (c). But by (c), $\nu_{n+1}^{\alpha_m} > \nu_{n+1}^{\alpha_m+1}$ for every $m < \omega$. Contradiction. ∎

If there is no inner model of $\exists \alpha o(\alpha) = \alpha^{++}$, then a sequence $\langle \tau_n | n < \omega \rangle$ of 2.1 is actually a sequence of indiscernibles for κ. This follows easily from Proposition 2.1 and the Mitchell Covering Lemma [Mi3].

PROPOSITION 2.2: *The final segment of the sequence $\langle \tau_n \mid n < \omega \rangle$ consists of indiscernibles for κ.*

Proof: Suppose otherwise. Then by the Mitchell Covering Lemma [Mi3] there is $h \in \mathcal{K}(\mathcal{F})$ and $\delta_n < \tau_n$ $(n < \omega)$ such that $h(\delta_n) \geq \tau_n$ for infinitely many n's. Define a club in $\mathcal{K}(\mathcal{F})$:

$$C = \{\nu < \kappa | h''(\nu) \subseteq \nu\}.$$

Then, by the choice of $\langle \tau_n | n < \omega \rangle$, there is $n_0 < \omega$ such that for every $n \geq n_0$ $\tau_n \in C$, which is impossible. Contradiction. ∎

3. On the strength of precipitousness of a nonstationary ideal over an inaccessible

We are going to show that the assumptions used in [Gi] making NS_κ precipitous $\left((\omega, \kappa^+ + 1)\text{-repeat point}\right)$ and $NS_\kappa^{\aleph_0}$ precipitous $\left((\omega, \kappa^+)\text{-repeat point}\right)$ over an inaccessible κ can be weakened to an $(\omega, \kappa + 1)$-repeat point and to an (ω, κ)-repeat point, respectively. This is quite close to the equiconsistency, since by [Gi], an $(\omega, < \kappa)$-repeat point is needed for the existence of such ideals.

THEOREM 3.1: *Suppose that there exists an $(\omega, \kappa+1)$-repeat point over κ. Then in a generic extension preserving inaccessibility of κ, NS_κ is a precipitous ideal.*

The proof combines constructions of [Gi] and [Gi1]. We will stress only the new points.

Sketch of the Proof: Let $\alpha < o(\kappa)$ be an $(\omega, \kappa+1)$-repeat point for $\langle \mathcal{F}(\kappa, \alpha') \mid \alpha' < o(\kappa) \rangle$, i.e. $\mathrm{cf}\,\alpha = \aleph_0$ and for every $A \in \bigcap\{\mathcal{F}(\kappa, \alpha^* + i) \mid i \leq \kappa\}$ there are unboundedly many β's in α such that $\beta + \kappa < \alpha$ and $A \in \bigcap\{\mathcal{F}(\kappa, \beta + i) \mid i \leq \kappa\}$.

As in [Gi] we first define the iteration $\mathcal{P}_\delta$ for δ in the closure of $\{\beta \leq \kappa \mid \beta$ is an inaccessible or $\beta = \gamma + 1$ for an inaccessible $\gamma\}$. On limit stages as in [Gi] the limit of [Gi2] is used. Define $\mathcal{P}_{\delta+1}$. If $o(\delta) \neq \beta + \delta$ or $o(\delta) \neq \beta + \delta + 1$ for some β then $\mathcal{P}_{\delta+1} = \mathcal{P}_\delta * C(\delta^+) * \mathcal{P}(\delta, o(\delta))$ exactly as in [Gi], where $C(\delta^+)$ is the Cohen forcing for adding δ^+ functions from δ to δ and $\mathcal{P}(\delta, o(\delta))$ is a forcing used in [Gi] for changing cofinalities without adding new bounded sets.

Now let $o(\delta) = \beta + \delta$ for some ordinal $\beta, \beta > \delta$. First we force as above with $C(\delta^+)$.

CASE 1: The value of the first Cohen function added by $C(\delta^+)$ on 0 is not 0.

Then we force as above with $\mathcal{P}(\delta, o(\delta))$.

CASE 2: The value of the first Cohen function added by $C(\delta^+)$ on 0 is 0.

Then we are going to shoot a club through $\bigcap\{\mathcal{F}(\delta, \beta + i) \mid i < \delta\}$ using the forcing notion Q described below.

$Q = \{\langle c, e\rangle \mid c \subseteq \delta$ closed, $|c| < \delta$, $e \subseteq \bigcap\{\mathcal{F}(\delta, \beta + i) \mid i < \delta\}$, $|e| < \delta\}$, $\langle c_1, e_1\rangle \leq \langle c_2, e_2\rangle$ iff c_2 is an end-extension of c_1, $e_1 \subseteq e_2$ and, for every $A \in e_1$, $c_2 \backslash c_1 \subseteq A$. Now every regular $i < \delta$ forcing with $\mathcal{P}(\delta, \beta + i)$ produces a club through $\bigcap\{\mathcal{F}(\delta, \beta + j) \mid j < i\}$ changing cofinality of δ to i. Thus Q contains an i-closed dense subset in any $\mathcal{P}(\alpha, \beta + i)$-generic extension of $V^{\mathcal{P}_\alpha * C(\alpha^+)}$. Based on this observation, we are going to use here the method of [Gi1]. It makes the iteration of such forcings Q possible.

If $o(\delta) = \beta + \delta + 1$ for some β, $\beta > \delta$, then we combine both previous cases together inside the Prikry sequence produced at this stage.

Namely, we proceed as follows. Let $i : V \to M \simeq \mathrm{Ult}(V, \mathcal{F}(\delta, \beta + \delta))$. We consider also the second ultrapower, i.e. $N \simeq \mathrm{Ult}(M, \mathcal{F}(i(\delta), i(\beta) + i(\delta)))$. Let $k : M \to N$ and $j = k \circ i : V \to N$ be the corresponding elementary embeddings.

Then, in N, $o(\delta) = \beta + \delta$ and $o(i(\delta)) = i(\beta) + i(\delta)$. So, in N, both δ and $i(\delta)$ are of the type of the previous cases. We want to deal with δ as in Case 1 and with $i(\delta)$ as in Case 2. This can easily be arranged, since we are free to change one value of a Cohen function responsible for the switch between Cases 1 and 2. The next stage will be to define an extension $\mathcal{F}^*(\delta, \beta+\delta)$ of $\mathcal{F}(\delta, \beta+\delta) \times \mathcal{F}(\delta, \beta+\delta)$ in $V[G_\delta]$, where $G_\delta \subseteq \mathcal{P}_\delta$ is generic. For this use [Gi1] where N was first stretched by using the direct limit of $\langle \mathcal{F}(i(\delta), i(\beta)+\xi) \mid \xi < i(\delta) \rangle$. Finally we force a Prikry sequence using $\mathcal{F}^*(\delta, \beta + \delta)$. Notice that the following holds:

(*) if $\langle \langle \delta_n, \rho_n \rangle \mid n < \omega \rangle$ is such a sequence then both $\langle \delta_n \mid n < \omega \rangle$ and $\langle \rho_n \mid n < \omega \rangle$ are almost contained in every club of δ of V.

Simply because $\langle \delta, i(\delta) \rangle \in j(C)$ for a club $C \subseteq \delta$ in V.

This completes the definition of $\mathcal{P}_{\delta+1}$ and hence also the definition of the iteration.

The intuition behind this is as follows. We add a club subset to every set $A \in \bigcap \{\mathcal{F}(\kappa, \alpha+i) | i \leq \kappa\}$. α is $(\omega, \kappa+1)$-repeat point, so A reflects unboundedly many times in α, i.e. $A \in \cap \{\mathcal{F}(\kappa, \beta + i) | i \leq \kappa\}$ for unboundedly many β's in α.

Reflecting this below κ, we will have $A \cap \delta \in \cap \{\mathcal{F}(\delta, \gamma + 1) | i \leq \delta\}$, where $o(\delta) = \gamma + \delta$. In [Gi, Sec. 3], we had $(\alpha, \kappa^+ + 1)$-repeat point which corresponds to $\bigcap \{\mathcal{F}(\delta, \gamma+i) | i \leq \delta^+\}$. Then just the forcing $\mathcal{P}(\delta, o(\delta))$ will add a club through every set in $\bigcap \{\mathcal{F}(\delta, \gamma + i) | i \leq \delta^+\}$. Here our assumptions are weaker and we use the forcing Q instead. There are basically two problems with this: iteration and integration with $\mathcal{P}(\delta, \beta)$'s. For the first problem the method of [Gi1] is used directly. The problematic point with the second is that once using Q we break the Rudin–Keisler ordering of extensions of $\mathcal{F}(\delta, \beta)$'s used in $\mathcal{P}(\delta, o(\delta))$. In order to overcome this difficulty, we split the case $o(\delta) = \beta + \delta$ into two. Thus in Case 1 we keep Rudin–Keisler ordering and in Case 2 force with Q. Finally, at stages α with $o(\delta) = \beta + \delta + 1$ both cases are combined in the fashion described above. The rest of the proof is as in [Gi, Sec. 3].

The following obvious changes needed to be made: instead of $E \in \bigcap \{\mathcal{F}(\kappa, \beta) | \alpha < \beta \leq \alpha + \kappa^+\}$ we now deal with $E \in \bigcap \{\mathcal{F}(\kappa, \beta) | \alpha < \beta \leq \alpha + \kappa\}$ and instead of $E(\kappa^+)$ there we use $E(\kappa) = \{\delta \in E |$ there is $\bar{\delta}$ s.t. $o^{\mathcal{F}}(\delta) = \bar{\delta} + \kappa$ and $\delta \cap E \in \bigcap \{\mathcal{F}(\delta, \delta') | \bar{\delta} \leq \delta' < \bar{\delta} + \kappa^+\}$ which belongs to $\mathcal{F}(\kappa, \beta + \kappa)$ for unboundedly many β's in α. Lemmas 3.2–3.5 of [Gi] have the same proof in the present context. The changes in the proof of Lemma 3.6 of [Gi] (actually the claim there) use the method of iteration of Q's and the principal (*).

If we are not concerned about a regular cardinal, then the same construction starting with an (ω,κ)-repeat point turns $\mathrm{NS}_\kappa^{\mathrm{Sing}}$ into a precipitous ideal. So the following holds:

THEOREM 3.2: *Suppose that there exists an (ω,κ)-repeat point over κ. Then in a generic extension preserving inaccessibility of κ, $\mathrm{NS}_\kappa^{\mathrm{Sing}}$ is a precipitous ideal.*

4. Open problems

1. Is the strength of $\mathrm{NS}_\kappa^{\aleph_0}$ precipitous over an inaccessible κ $(\omega,<\kappa)$-repeat point?
2. Can a model for NS_κ precipitous over an inaccessible κ be constructed from something weaker than an $(\omega,\kappa+1)$-repeat point?
3. What is the strength of NS_κ precipitous over the first inaccessible?

The upper bound for (3) is a Woodin cardinal, see [Sh-Wo]. If it is possible to construct a model with $\mathrm{NS}_\kappa^{\aleph_0}$ precipitous from an $(\omega,<\kappa)$-repeat point, then we think that this assumption is also sufficient for (2) and (3).

4. How strong is "there is a precipitous ideal over the first inaccessible"?

By [Sh-Wo] a Woodin cardinal suffices. On the other hand, one can show that at least $o(\kappa)=\kappa$ is needed.

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